Practice Find the derivative.

① $f(x) = (x^2+3x)\sin(x)$

② $G(p) = \dfrac{\sin(p)}{p^2+3p}$

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① $f'(x) = (x^2+3x)'\sin(x) + (x^2+3x)(\sin(x))'$

$= \boxed{(2x+3)\sin(x) + (x^2+3x)\cos(x)}$

② $G'(p) = \boxed{\dfrac{Ho\,dHi - Hi\,dHo}{Ho\,Ho}} = \boxed{\dfrac{(p^2+3p)\cos(p) - \sin(p)(2p+3)}{(p^2+3p)^2}}$

$\left(\dfrac{Hi}{Ho}\right)' \rightarrow$

More examples

③ What is the formula for the derivative of $\cot(\theta)$?

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solution: $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

$\Rightarrow [\cot(\theta)]' = \dfrac{Ho\,dHi - Hi\,dHo}{Ho\,Ho} = \dfrac{\sin(\theta)\cdot[-\sin\theta] - \cos(\theta)[\cos\theta]}{\sin^2(\theta)}$

$= \dfrac{-\sin^2(\theta) - \cos^2(\theta)}{\sin^2(\theta)} = \boxed{\dfrac{-1}{\sin^2(\theta)}} = \boxed{-\csc^2(\theta)}$

4 Find the derivative of

$$\frac{x^2\cos(x)}{x^3+\sqrt{x}}.$$

Type some Sage code below and press Evaluate.

```
f(x) = x^2*cos(x)/(x^3+sqrt(x))
show(diff(f(x),x))
show(diff(f(x),x).factor())
```

Evaluate

$$-\frac{\left(6\,x^2+\frac{1}{\sqrt{x}}\right)x^2\cos(x)}{2\left(x^3+\sqrt{x}\right)^2}-\frac{x^2\sin(x)}{x^3+\sqrt{x}}+\frac{2\,x\cos(x)}{x^3+\sqrt{x}}$$

$$-\frac{\left(2\,x^{\frac{7}{2}}\sin(x)+2\,x^{\frac{5}{2}}\cos(x)+2\,x\sin(x)-3\,\cos(x)\right)x^{\frac{3}{2}}}{2\left(x^3+\sqrt{x}\right)^2}$$

$$\left[\frac{x^2\cos(x)}{x^3+\sqrt{x}}\right]' = \frac{(x^3+\sqrt{x})\left[x^2\cos(x)\right]' - x^2\cos(x)\left[x^3+\sqrt{x}\right]'}{(x^3+\sqrt{x})^2}$$

$$\frac{\text{HodHi}-\text{HiodHo}}{\text{HoHo}}$$

$$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{(x^3+\sqrt{x})\left(x^2(-\sin(x))+2x\cos(x)\right) - x^2\cos(x)\left\{3x^2+\frac{1}{2\sqrt{x}}\right\}}{(x^3+\sqrt{x})^2}$$

obviously, this is the same as the sagemath answer.

Another way answers can look different.

Olivia's answer : $\cos(2x)+\frac{1}{2}$

Alexa's answer : $2\cos^2(x)-\frac{1}{2}$.

Actually the same

Because $\cos(2x)+\frac{1}{2} = (2\cos^2(x)-1)+\frac{1}{2}$

$= 2\cos^2(x)-\frac{1}{2}$.

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5 Find the derivative of $\sec(x)$.

$$\sec(x) = \frac{1}{\cos(x)}$$

$$(\sec(x))' = \frac{\cos(x)\cdot[1]' - 1\cdot[\cos(x)]'}{(\cos(x))^2}$$

$$(\sec(x))' = \frac{\cos(x)\cdot 0 - 1\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot 0 - 1\cdot(-\sin(x))}{\cos(x)\cdot\cos(x)}$$

$$= \frac{\sin(x)\cdot 1}{\cos(x)\cdot\cos(x)} = \boxed{\frac{\sin(x)}{\cos^2(x)}}$$

$$= \tan(x)\cdot\sec(x) = \boxed{\sec(x)\cdot\tan(x)}.$$

The Chain Rule.

Background: Composition of Functions

$f(x) = \sqrt{x}$

$g(x) = \cos(x)$

$(f \circ g)(x) = f(g(x)) = f(\cos(x)) = \sqrt{\cos(x)}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \cos(\sqrt{x})$.

$u(x) = 2x$

$v(x) = 3x - 1$

$(u \circ v)(x) = u(3x-1) = 2(3x-1) = 6x-2$

$(v \circ u)(x) = v(2x) = 3(2x) - 1 = 6x-1$

different.

"Composition of Functions is not commutative."

Chain Rule.

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

$$[f(\text{Bubba})]' = f'(\text{Bubba}) \cdot (\text{Bubba})'$$

Proof:

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{(g(x+h) - g(x))} \left(\frac{h}{(g(x+h) - g(x))}\right)$$

$$= \lim_{h\to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{(g(x+h) - g(x))}}_{f'(g(x))} \cdot \underbrace{\left(\frac{g(x+h) - g(x)}{h}\right)}_{g'(x)} = f'(g(x)) \cdot g'(x).$$

$\blacksquare$

Examples

① $[\sin(3\theta + \frac{\pi}{2})]'$

$= \sin'(3\theta + \frac{\pi}{2}) \cdot (3\theta + \frac{\pi}{2})'$

$= \cos(3\theta + \frac{\pi}{2}) \cdot 3$

$= \boxed{3\cos(3\theta + \frac{\pi}{2})}$

$[f(\underbrace{g(\theta)}))]'$

inside fcn $= 3\theta + \frac{\pi}{2}$

outside fcn. $\sin(x)$.

② $[x \cdot e^{x^5}]'$

$= \underline{1} \cdot e^{x^5} + x \cdot [e^{x^5}]'$

outside: e^x
inside $= x^5$

$(e^x)' = e^x$

$= e^{x^5} + x \cdot \underbrace{e^{x^5}}_{f'(g(x))} \cdot [x^5]'$

$g(x) = x^5$
$f(x) = e^x$
$f'(g(x)) = e^{g(x)}$
$= e^{x^5}$

$= e^{x^5} + x \cdot e^{x^5} \cdot 5x^4$

$= \boxed{e^{x^5} \cdot (1 + 5x^5)}$

③ $[\sin(\cos(\sin(A)))]' = ?$

$= \sin'(\cos(\sin(A))) \cdot [\cos(\sin(A))]'$

$= \cos(\cos(\sin(A))) \cdot \cos'(\sin(A)) \cdot \sin'(A)$

$= \cos(\cos(\sin A)) \cdot (-\sin(\sin(A)) \cdot \cos(A))$

$= \boxed{-\cos(\cos(\sin A)) \cdot \sin(\sin(A)) \cdot \cos(A)}$

Type some Sage code below and press Evaluate.

```
f(A)=sin(cos(sin(A)))
show(diff(f(A),A))
```

Evaluate

$$-\cos(A)\cos(\cos(\sin(A)))\sin(\sin(A))$$

Quiz

(1) What is your name?

(2) What is your favorite music group? ← individuals are OK.

(3) Simplify

(a) $$\frac{x^2+\sqrt{x}}{3\sqrt{x}}$$

(b) $\sin(2x)\sec(x)$.

(c) Your life.

(4) Find the derivative

$$g(x) = 2x\cos(x) + 5e^x - 7.$$